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Geometry and Topology of Local Minimal 2-Trees

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Abstract. The aim of this paper is to get an effective restriction on the topologies of minimal 2-trees in terms of twisting numbers of the trees and the convexity levels number of the trees boundaries.

Introduction

The main aim of this paper is to get some effective restrictions for the possible topologies of nondegenerate (full) planar local minimal networks spanning a fixed finite set M of points of the plane. We obtained such restriction in terms of the number of convexity levels of the set M. This result could be used for optimization of well-known algorithms enumerating such networks to find the Steiner minimal tree.

This work is a part of the great branch of modern mathematics which is devoted to investigation of *Steiner Minimal Networks* which are solutions of the *Steiner Problem* or *One-Dimensional Plateau Problem.* A lot of papers devoted to this problem in its various forms [**1**]–[**21**], [**34**]–[**42**], see also [**22**]–[**33**].

There are different opinions regarding who is the author of the problem. However, the most versions are wrong, and there are very simple reasons explaining these mistakes. We decided to enclose a short historical review on the Steiner Problem based on the works of M. Zacharias [**30**] and H. W. Kuhn [**31**] which we have found in the book [**16**] of F. K. Hwang, D. Richards and P. Winter. See also [**34**].

The first version of the Steiner problem appeared long before Steiner

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in the works of Fermat (1601–1665): *find in the plane a point S, the sum of whose distances from three given points A, B and C is minimal.* Before 1640, Torricelli had proposed the following geometrical solution to this problem. Let us construct on the sides of the triangle ABC three equilateral triangles ABC', BCA', and CAB', lying completely outside ABC. Circumscribe the circles around each triangle. Then, Torricelli asserted, these three circles intersect at the single point that is the solution S of the Fermat Problem. This intersection point is called *Torricelli point.*

In 1647, Cavalieri found an important property of the Torricelli point. It turns out that the angles between the segments joining the Torricelli point with the vertices of the triangle ABC are equal to each other and, hence, equal to 120°.

In 1750, Simpson found another way to construct the Torricelli point. Let us draw equilateral triangles on the sides of the triangle ABC as before. Then, the three segments AA', BB', and CC' pass through one point, which is just the Torricelli one. These segments are called *Simpson lines.*

In 1834 Heinen, and also in 1853 Bertrand, observed that the solutions of the Fermat problem which we described above are not correct in general. Really, the approaches of Torricelli and Cavalieri work if and only if all angles of the triangle ABC are at most 120°. If one of these angles, say A, is greater than 120°, then the Torricelli point lies outside the triangle ABC and cannot be the solution of the Fermat Problem. In this case the solution is the point A. Also, in the same 1834, Heinen observed that if the angles of the ABC are at most 120°, then the lengths of the three Simpson lines are equal to each other and equal to the sum of distances from the Torricelli point to the vertices of the triangle ABC.

Besides the solution of the Fermat Problem mentioned above, Simpson stated some generalization of that problem: *find in the plane (in d-dimensional Euclidean space) a point, the sum of whose distances from n given points is minimal.* This problem appeared as an exercise in Simpson's book "*Fluxions*" and attracted the attention of many well-known mathematicians including Steiner.

Another important generalization of the Fermat Problem was proposed by Jarnik and Kössler [**32**] in 1934, namely, *find a shortest network spanning n points in the plane.* Jarnik and Kössler found a shortest network spanning the vertices of a regular n-gon for $n = 3,\ 4,\ 5$, and, besides that, they proved that for $n \geq 13$ such shortest networks consist of $n-1$ sides of the polygon. Jarnik and Kössler made no reference to the Fermat Problem since their own problem seemed quite different.

In the book "*What is Mathematics?*", published in 1941, R. Courant and H. Robbins discussed the problem of Jarnik and Kössler, but called it the Steiner Problem. The authors did not refer to either Fermat for the case $n = 3$ or to Jarnik and Kössler for the general n case. Due to the popularity of that book, the terminology "Steiner Problem" has been generally adopted. The classic work by Courant and Robbins has originated not only a mistake in the attribution of priority but, more importantly, a great deal of interest in this problem.

Now, let us formulate the main result of the present work. To do that, recall the well-known notion of convexity levels of a finite subset of the plane [**33**], and the important geometrical characteristic of a planar binary tree, the twisting number, which we have introduced in our previous papers [**34,35,36,37,38,39**].

Let M be a finite set of points in the plane. Put into the *first convexity level* M^1 all points from M lying on the boundary of the convex hull of the set M. Note that the set M^1 is not empty. Throw M^1 out of the set M. If the obtained set is not empty, use the same procedure to reconstruct this new set. Namely, consider the convex hull of the set $M \setminus M^1$ and put all points lying on the boundary of this convex hull into the *second convexity level* M^2. Continue this process until the set M will be exhausted. The obtained partition $M = \sqcup M^i$ is called the *partition into convexity levels*, and the set M^i itself is called the *ith convexity level for the set* M. If $M = \sqcup_{i=1}^k M^i$ then we say that *M has k convexity levels.* Note that the set M lying on the boundary of its convex hull has just one convexity level $M^1 = M$.

Further, let Γ be a planar binary tree (in what follows we shall refer such trees as *2-trees*), that Γ is a planar tree and the degrees of its vertices are supposed to be either one or three. Let a and b be arbitrary edges of Γ, and $\gamma[a,b]$ be the unique path in Γ joining a and b. Then, during the "motion" along $\gamma[a,b]$ from a to b, at each inner vertex of $\gamma[a,b]$ we "turn" either to the left or to the right. The difference between the total numbers of the left and the right turns is called the *twisting number* $\mathrm{tw}(a,b)$ *of the ordered pair* (a,b). We put $\mathrm{tw}(a,a)=0$ for any edge a of Γ.

Definition. The *twisting number* $\mathrm{tw}\,\Gamma$ *of a planar 2-tree* Γ is the maximum of the twisting numbers over all ordered pairs of its edges:

$$\mathrm{tw}\,\Gamma = \max_{(a,b)}\ \mathrm{tw}_\Gamma(a,b).$$

Now we can formulate the main result of this work.

The Main Theorem. *If a planar local minimal 2-tree* Γ *spans a finite subset* M *of the plane, and* M *has* k *convexity levels, then the twisting number* $\mathrm{tw}\,\Gamma$ *does not exceed* $12(k-1)+5$. *Moreover, this estimation is exact: for any* k *there exists a minimal 2-tree* Γ, $\mathrm{tw}\,\Gamma = 12(k-1)+5$, *spanning a set* M *with exactly* k *convexity levels.*

Note that The Main Theorem restricts strongly the set of possible topologies of local minimal 2-trees in terms of the convexity levels number of the boundary set.

The Main Theorem generalizes the following result of the authors [**34,35,36**].

Corollary. *The twisting number of a planar minimal 2-tree with convex boundary does not exceed 5.*

Note that in the case of convex boundaries the converse result is true: *if the twisting number of a planar 2-tree* Γ *does not exceed five, then there exists a minimal 2-tree planar equivalent to* Γ *and with a convex boundary*, see [**34**]–[**39**]. Unfortunately, we don't know if the converse statement of The Main Theorem true in general situation.

1. Polygonal lines

To start with, we need to prove some special properties of polygonal lines. First, recall the definition of a polygonal line.

Definition. A finite set of nondegenerate segments $\Delta_i = [A_i, A_{i+1}]$, $i = 0, \dots, n$, is called *an unclosed polygonal line* if $\Delta_i \cap \Delta_j \neq \varnothing$, $0 \leq i < j \leq n$, implies $j = i + 1$ and $\Delta_i \cap \Delta_j = \{A_{i+1}\}$. A finite set of nondegenerate segments $\Delta_i = [A_i, A_{(i+1) \bmod n}]$, $i = 0, \dots, n-1$, is called *a closed polygonal line* if $\Delta_i \cap \Delta_{j \bmod n} \neq \varnothing$, $0 \leq i < j \leq n$, $i \neq j \bmod n$, implies $j = (i+1) \bmod n$ and $\Delta_i \cap \Delta_{j \bmod n} = \{A_{(i+1) \bmod n}\}$.

The segments Δ_i are called *the edges* of L, and the points A_i are called *the vertices* of L. If L is unclosed, then the vertices A_0 and A_{n+1} are called *ending vertices*, and all other vertices are called *inner* ones. Also, if L is unclosed, then the edges Δ_0 and Δ_n are called the *ending edges*, and all other Δ_i are called the *inner edges*.

Let L be a polygonal line. By definition, its vertices are canonically enumerated. Clearly, we can enumerate the vertices in the converse way, and we also obtain a polygonal line. Each of the corresponding two orderings of the polygonal line vertices is called *the orientation* of L. If L is unclosed oriented polygonal line then we call its first ending vertex the *beginning* one and the last ending vertex the *ending* one. If an orientation of a polygonal line L is chosen, then we can consider each edge Δ_i as vector A_iA_{i+1} for unclosed L, and as vector $A_iA_{(i+1) \bmod n}$ for closed L.

Let (a, b) be an ordered pair of non-zero vectors on the plane. Suppose that these vectors have nonopposite directions. Thus, we can define *an oriented angle* $\alpha(a, b)$ *from* a *to* b as follows. The absolute value of $\alpha(a, b)$ is equal to the least angle between a and b, and if a and b are linear independent then the sign of $\alpha(a, b)$ equals the sign of the oriented frame (a, b). In what follows, it will be more useful to consider the normalized oriented angles, namely, we shall call the value $\alpha(a, b)/(\pi/3)$ *the twisting from the vector* a *to the vector* b and denote it by $\operatorname{tw}(a, b)$.

Let L be an oriented unclosed polygonal line whose consecutive edges are $a_i = [A_i, A_{i+1}]$, $i = 0, \dots, n$.

Definition. The twisting $\mathrm{tw}(a_{i-1}, a_i)$ is called *the twisting at the vertex* A_i and is denoted by $\mathrm{tw}\, A_i$. The sum of twistings over all inner vertices of L is called *the turning along* L and is denoted by $\mathrm{tn}\, L$:

$$\mathrm{tn}\, L = \sum_{i=1}^{n} \mathrm{tw}\, A_i.$$

Notice the following properties of the turning:

- if we change the orientation of the polygonal line L then the turning changes the sign (skew-symmetry);
- if P is an arbitrary point lying inside an edge of an unclosed polygonal line L, and L_1 and L_2 are the polygonal lines which P partitions L into, and if the orientations of the both L_i are induced from L, then $\mathrm{tn}\, L = \mathrm{tn}\, L_1 + \mathrm{tn}\, L_2$ (additivity along a path).

Let L be an oriented closed polygonal line with vertices A_i, $i = 0, \dots, n-1$. We define the *twistings* at its vertices just in the same way.

Definition. The sum of twistings over all vertices of a closed polygonal line L is called *the turning along* L and is denoted by $\mathrm{tn}\, L$:

$$\mathrm{tn}\, L = \sum_{i=0}^{n-1} \mathrm{tw}\, A_i.$$

The following proposition is well-known.

Proposition 1.1. *If L is a closed polygonal line oriented anticlockwise then* $\mathrm{tn}\, L = 6$. *If L is oriented clockwise then* $\mathrm{tn}\, L = -6$.

Let L be a polygonal line. Suppose that L' is an other polygonal line coinciding with L as subset of the plane, but having some additional vertices. Such polygonal line L' is called *a subdivision of* L. Clearly, the turning of an arbitrary subdivision of a polygonal line L equals the turning of L.

In what follows, we shall need the notion of deformation of a polygonal line.

Definition. A family L^t, $t \in [0,1]$, of polygonal lines is called *a deformation of the polygonal line* $L = L^0$ if the vertices A_i^t of L^t form continuous curves A_i^t parameterized by t.

The following proposition is evident.

Proposition 1.2. *Let L^t be a deformation of a polygonal line L. If L is unclosed, then suppose that during this deformation the directions of the ending edges of L remain fixed. Then the turning also remains fixed:*

$$\operatorname{tn} L^t = \text{const} = \operatorname{tn} L, \qquad t \in [0,1].$$

Let L^1 and L^2 be two polygonal lines.

Definition. We say that L^1 and L^2 *are in general position* if they intersect each other by at most finite number of points, and no one of the intersection points is a vertex of L^1 or L^2. If we are given that some vertices of L^1 and L^2 *are fixed*, then we say that these L^i *are in general position* if, again, they intersect each other by at most finite number of points all of which, except the fixed ones, are not the vertices of L^i, $i = 1,2$.

Let L^1 and L^2 be two oriented polygonal lines being in general position. Consider the set $L^2 \setminus (L^1 \cap L^2)$ and denote the closures of the connected components of this set by L_j^2. Each L_j^2 is a polygonal line which we orient in correspondence with the orientation of L^2. Let $A_0^2, A_1^2, \ldots$ be the consecutive vertices of the polygonal line L^2. Let us suppose that the polygonal lines L_j^2 are enumerated in correspondence with the orientation of L^2 and such that L_1^2 contains A_0^2. Moreover, if A_0^2 is an ending point of L_1^2, then we suppose that A_0^2 is the beginning vertex of the oriented polygonal line L_1^2.

Definition. The partition $L^2 = \cup L_j^2$ is called *the canonical partition of L^2 with respect to L^1.*

Now, let L^1 and L^2 be unclosed polygonal lines being in general position and joining the same ending vertices A and B which are supposed to be fixed. Orient L^1 and L^2 from A to B. Let $L^2 = \cup_{j=1}^N L_j^2$ be the canonical partition of L^2 with respect to L^1. The elements L_1^2 and L_N^2

are called *the first* and *the last elements*, respectively, or *the ending elements* of the partition, and all other L^2_j are called *inner elements.* The orientation of L^2 defines the *positive direction of motion* along L^2 and, thus, along each L^2_j.

Note that each L^2_j and L^1 bound an open domain Ω_j. More precisely, if A_j and B_j are the ending vertices of L^2_j, then denote by b_j the part of L^1 between A_j and B_j. The domain Ω_j is the one bounded by closed polygonal line $L^2_j \cup b_j$. The polygonal line b_j is called *the base of* Ω_j. Sometimes we will refer to the domains Ω_j as Ω*-domains.* We call the first and the last Ω_j *the ending* Ω*-domains*, and the other Ω_j *the inner* Ω*-domains.*

Definition. Define *the sign* $\operatorname{sign}(L^2_j)$ *of* L^2_j to be equal $+1$ if the anticlockwise motion along the boundary $\partial\Omega_j$ of Ω_j induces the motion along L^2_j in the positive direction. Otherwise, we put $\operatorname{sign}(L^2_j) = -1$. The sum of $\operatorname{sign}(L^2_j)$ over all $j = 1, \dots, N$ is called *the index of* L^2 *with respect to* L^1 and is denoted by $\operatorname{Ind}(L^2, L^1)$.

Consider the ending domains Ω_1 and Ω_N, and orient their boundaries $\partial\Omega_1$ and $\partial\Omega_N$ with respect to the orientation of L^2. Denote by $\alpha(L^2, L^1)$ the twisting of $\partial\Omega_1$ at A, and by $\beta(L^2, L^1)$ the twisting of $\partial\Omega_N$ at B. We call these twistings *the first* and *the last twistings of* L^2 *with respect to* L^1.

Now we can formulate the main theorem of this section.

Theorem 1.1. *Let* L^1 *and* L^2 *be two unclosed polygonal lines joining some points* A *and* B *and oriented from* A *to* B. *Suppose that* A *and* B *are fixed and these polygonal lines are in general position. Then we have:*

$$\operatorname{tn} L^2 = \operatorname{tn} L^1 + 6\ \operatorname{Ind}(L^2, L^1) - \alpha(L^2, L^1) - \beta(L^2, L^1).$$

Before proving the theorem, let us discuss some other geometrical definition of the index defined above.

Let L^1 and L^2 be as above. Let us partition the set of the inner domains and the set of the corresponding parts of L^2 in the following way.

Definition. An inner domain Ω_j and the corresponding L_j^2 are called *A-domain and A-element*, respectively, if Ω_j contains A and does not contain B. If it contains B but does not contain A, then Ω_j and L_j^2 are called *B-domain* and *B-element.* If Ω_j contains the both A and B, then it and L_j^2 are called *full*, or *F-domain and F-element*, respectively, and if A and B lie outside Ω_j, then it and L_j^2 are called *empty*, or *E-domain and E-element.*

Now, we expand the partition constructed above to the ending domains and the corresponding ending elements of L^2.

Definition. If there exist at least two Ω-domains, then the first one is called *A-domain*, and the last one *B-domain.* If there exists just one Ω-domain, then we call it *A-domain.*

Define *the sign of* Ω_j to be the $\mathrm{sign}(L_j^2)$ and denote it by $\mathrm{sign}(\Omega_j)$.

Now, assign to each A-domain Ω_j the letter $A^{\mathrm{sign}(\Omega_j)}$, to each B-domain Ω_j the letter $B^{\mathrm{sign}(\Omega_j)}$, to each full domain Ω_j the letter $F^{\mathrm{sign}(\Omega_j)}$, and to each empty domain Ω_j the letter $E^{\mathrm{sign}(\Omega_j)}$. Generate *the word* $W(L^2, L^1)$ by writing consequently the letters corresponding to consecutive domains. We define *the weight* $\mathrm{weight}(L^2, L^1)$ of the word $W(L^2, L^1)$ to be equal to the sum of powers over all letters of the word $W(L^2, L^1)$. From the definition we immediately obtain the following corollary.

Corollary 1.1. *Under the above notations, we have*

$$\mathrm{Ind}(L^2, L^1) = \mathrm{weight}(L^2, L^1).$$

Now, let us prove Theorem 1.1.

Proof. The idea of the proof is as follows. First, we prove that some subdivision of the polygonal line L^2 can be deformed to a polygonal line with the same ending edges and such that the resulting polygonal line has no empty domains (with respect to L^1). By Proposition 1.2, the turning along the obtained polygonal line equals the turning along L^2. Further, we show that this deformation can be represented as a consequence of so-called elementary transformations, and each the elementary transformation does not change the index $\operatorname{Ind}(L^2, L^1)$. At last, we prove Theorem 1.1 for polygonal lines L^2 which have no empty domains.

To start with, consider an arbitrary empty domain Ω_i, and suppose that there exists a point $P \in L^2$ in the interior of the base b_i of Ω_i. Since L^2 and L^1 are in general position, L^2 should come inside Ω_i, and, since Ω_i is empty, L^2 should also come out of Ω_i through some other point P' lying in the interior of the base b_i. Thus, in this case there exists another empty domain Ω_j such that $\Omega_j \subset \Omega_i$.

Consider a maximal chain of Ω-domains of the following form:

$$\Omega_i \supset \Omega_{i_1} \supset \cdots \supset \Omega_{i_k}.$$

It is clear that inside the base of Ω_{i_k} and, thus, inside Ω_{i_k} there are no other points of L^2.

Definition. An empty Ω-domain which does not contain any other point of L^2 inside it and, thus, inside its base, is called *pure empty domain* and such bases are called *pure bases.*

Thus, we have proven the following lemma.

Lemma 1.1. *If the set of empty Ω-domains is not empty, then among such domains there exists a pure empty Ω-domain.*

The next evident lemma will be useful.

Lemma 1.2. *Let W be a closed polygonal line, A and B are some its vertices, and l and b are two parts of W lying between A and B. Then there exist a deformation l^t of some subdivision of l to some subdivision of b such that the points A and B remain fixed and all polygonal lines l^t lie in the closed domain bounded by W.*

Let Ω_i be a pure empty domain. Construct a deformation of some subdivision of L_i^2 to some subdivision of b_i described in Lemma 1.2. This deformation generates deformation of the corresponding subdivision of L^2 and will be used as the first part of the elementary transformation. We denote by $\bar{L}^2$ the resulting polygonal line.

To obtain the last part of the transformation, let us construct at each vertex of the subdivision of the base $b_i \subset \Omega_i$ described in Lemma 1.2, a unit vector directed out of the Ω_i. Then, the last part of the elementary transformation is the following deformation of b_i extended to the deformation of $\bar{L}^2$. Let the vertices of the constructed subdivision of b_i go along these unit vectors. This motion of the vertices can be extended evidently to a deformation of $\bar{L}^2$. If the vertices go not far from the previous position of b_i, new intersections with L^1 do not occur, but the intersection along b_i disappeares.

Thus, combining these two deformations described above we reconstruct the polygonal line L^2 in such a way that the number of intersection points with L^1 is decreased by 2.

Definition. The composition of two deformations described above is called *an elementary transformation* of the polygonal line L^2.

It is clear that after a few elementary transformations the reconstructed polygonal line L^2 won't have empty domains. The reconstructed polygonal line L^2 obtained in such a way is said to be *reduced.* It is clear that the elementary transformations preserve the property of the polygonal line L^2 to be in general position with respect to the polygonal line L^1. Thus, the reduced polygonal line L^2 is also in general position with respect to L^1.

Now, let us prove the following lemma.

Lemma 1.3. *An elementary transformation does not change the index of L^2 with respect to L^1.*

Proof. Consider an elementary transformation which deletes an empty pure domain Ω_i. As a result, we reconstruct the three consecutive domains Ω_{i-1}, Ω_i, and Ω_{i+1} and obtain a new domain Ω'.

Consider the inverse transformation to this elementary one. The first part of the inverse transformation is the inverse transformation to last part of the elementary one. We have two different cases for the first part of the inverse transformation.

1. During this deformation the domain Ω' splits into two domains. It can be if a part of the boundary of Ω' touches from inside of Ω' either itself, or a part of L^1 lying inside Ω'.
2. Some domain Ω'' outside Ω' appears, and the deformed Ω' is reconstructed to a domain by adding to it the domain Ω''. It happens

when a part of boundary of Ω' touches from outside of Ω' either another part of the boundary, or a part of L^1 lying outside Ω'.

In the first case the both domains onto which Ω' splits have the same signs as Ω' has. The empty domain which appears during the last part of the inverse transformation has, evidently, the opposite sign. Thus, the index is preserved in this case.

In the second case the outer domain Ω'' has the opposite sign to the sign of Ω', but the reconstructed Ω', of course, has the same sign as Ω' has. The empty domain, evidently, also has the same sign as the Ω' has. Thus, again, we have that the index is preserved. Lemma 1.3 is proven.

Now, it remains to show that Theorem 1.1 holds for polygonal lines without empty domains. Consider an arbitrary inner intersection point P of L^1 and L^2. Let L^2_i and L^2_{i+1} be the consecutive parts of the canonical partition, the first of which finishes at P, and the second one starts at P. Let Q and R be the other boundary points of the parts L^2_i and L^2_{i+1}, respectively.

Definition. The point P is called *the point of monotonicity* if the points Q, P, and R are consecutive ones in the order induced by the orientation of the polygonal line L^1.

Now we formulate some properties of the points of monotonicity. Let Ω_{i-1} and Ω_i be two consecutive domains, and suppose that the ending point of L^2_{i-1} coinciding with the beginning point of L^2_i is point of monotonicity.

Lemma 1.4. *Under the above assumptions, we have*

- *if* Ω_i *is* A*-domain, then* Ω_{i-1} *is either* A*-domain, or* E*-domain, and* $\Omega_{i-1} \subset \Omega_i$;
- *if* Ω_i *is* F*-domain, then* Ω_{i-1} *is either* A*-domain, or* E*-domain, and* $\Omega_{i-1} \subset \Omega_i$;
- *if* Ω_{i-1} *is* F*-domain, then* Ω_i *is either* B*-domain, or* E*-domain, and* $\Omega_{i-1} \supset \Omega_i$;
- *if* Ω_{i-1} *is* B*-domain, then* Ω_i *is either* B*-domain, or* E*-domain, and* $\Omega_{i-1} \supset \Omega_i$.

Lemma 1.5. *Suppose that* L^2 *has no empty domains. Then all its inner* *intersection points with* L^1 *are points of monotonicity.*

Proof. Suppose otherwise, and consider the first domain Ω_i such that one of the ending points of the corresponding element L_i, say P, is not the point of monotonicity.

Let Ω_i be A-domain. Then, by Lemma 1.4, we have

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_i,$$

and all these Ω_j are A-domains.

This implies that the element L^2_{i+1} should come to the base of Ω_i from outside of Ω_i, thus the polygonal line L^2 comes into Ω_i through some point $\bar{P}$ belonging to the interior of the base of Ω_i. Since Ω_i does not contain the point B, the polygonal line L^2 should return back to some point $\bar{Q}$ of the interior of the base. Denote the part of L^2 between $\bar{P}$ and $\bar{Q}$ by $\bar{L}$.

Recall that all domains before Ω_i are A-domains. Let Ω' be the A-domain with the least number into which the path $\bar{L}$ comes. Let L' be the element of the canonical partition of L^2 corresponding to Ω'. Denote by L'' a connected part of $\bar{L}$ inside Ω', by A'' and B'' the ending points of L'', and by A' and B' the ending points of L'. To be definite, suppose that the point A' is closer to A than the point B' in the order along L^1.

In what follows, let us write $L[A,B]$ for the part of a nonclosed polygonal line L lying between its points A and B. If we throw out a point A or B, we indicate it by writing the corresponding "(" or ")" instead of "[" or "]", for example, $L(A,B)$ denote the part of L between A and B excluding the both A and B.

Now, note that $L^1[A'',B''] \subset L^1[A',B']$.

The path L'', together with $L^1[A'',B'']$, bounds an Ω-domain Ω''. The domain Ω'' can not contain the point A. Suppose otherwise. Since the point $A' \in \partial\Omega'$ does not belong to $L^1[A'',B''] = \partial\Omega' \cap \partial\Omega''$, and $\Omega'' \subset \Omega'$, then the point A' lies outside Ω''. Thus, the $L^1[A,A']$ should intersect the boundary of Ω''. But $L^1[A,A'] \cap L^1[A'',B''] = \varnothing$, and $L'' \cap L^1[A,B] = \{A'',B''\}$, that is $L'' \cap L^1[A,A'] = \varnothing$, contradiction. Thus, we have constructed an empty domain, contradiction.

Let Ω_i be an F-domain. By Lemma 1.4,

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_i,$$

and all these Ω_j, $j < i$, are A-domains.

Let A_i and B_i be the first and the last points of L_i^2. If L_{i+1}^2 comes to $L^1(A_i, B_i)$, then we obtain E-domain by the similar reasons as before. If it comes to $L^1(A, A_i)$, then it comes to the boundary of some A-domain from outside of it, and we can apply just the same reasoning as above.

So, it remains to consider the last case, namely, when Ω_i is a B-domain. If again we denote by A_i and $B_i = P$ the first and the last points of L_i^2, then it is easy to see that L_{i+1}^2 should come back to $L^1(A_i, B_i)$, and we obtain E-domain. The last contradiction completes the proof. □

Lemmas 1.4 and 1.5 give us the following important Corollary.

Corollary 1.2. *If L^1 and L^2 are two polygonal lines joining the same points A and B, being in general position, and such that L^2 does not have empty elements, then the word $W(L^2, L^1)$ can be only of one of the following forms:*

- *$A^pF^qB^r$, where p, q, and r are simultaneously either positive, or negative, and $q = \pm 1$, or*
- *A^pB^q, where $pq \le 0$, $p \ne 0$.*

Now, consider an arbitrary domain Ω_i. Let us move around it in the direction induced by the orientation of L^2, and use Proposition 1.1. Let A_i and B_i be the first and the last points of L_i^2, then we have

$$\operatorname{tn} L_i^2 - \operatorname{tn} L_i^1 + \alpha_i(L_i^2, L_i^1) + \beta_i(L_i^2, L_i^1) = 6 \ \operatorname{sign} \Omega_i, \qquad ((*_i))$$

where $\alpha_i(L_i^2, L_i^1)$ and $\beta_i(L_i^2, L_i^1)$ are the first and the last twistings of L_i^2 with respect to L_i^1.

Now, observe that the monotony of inner intersection points implies that

$$\beta_i(L_i^2, L_i^1) = -\alpha_{i+1}(L_{i+1}^2, L_{i+1}^1).$$

Then, summing the equations $(*_i)$ over all admissible i, we obtain

$$\operatorname{tn} L^2 - \operatorname{tn} L^1 + \alpha(L^2, L^1) + \beta(L^2, L^1) = 6 \ \operatorname{Ind}(L^2, L^1).$$

Theorem 1.1 is proven. □

Theorem 1.1, Corollary 1.1, Corollary 1.2, and the ideas of the proof of Lemma 1.3 gives us some useful estimation of the turning along the polygonal line L^2 in terms of the number of the A- and B-domains.

Definition. The number of all A- and B-elements of the canonical partition of the polygonal line L^2 with respect to the polygonal line L^1 is called *AB-module of L^2 with respect to L^1* and is denoted by $\mathrm{ABMod}(L^2, L^1)$. The number $\mathrm{ABMod}(L^2, L^1) + 1$ is called *the module of L^2 with respect to L^1* and is denoted by $\mathrm{Mod}(L^2, L^1)$.

Corollary 1.3. *Under the above notations and the assumptions of Theorem 1.1, we have*

$$|\,\mathrm{Ind}(L^2, L^1)| = |\,\mathrm{weight}(L^2, L^1)| \le \mathrm{Mod}(L^2, L^1),$$

thus

$$|\,\mathrm{tn}\, L^2| < |\,\mathrm{tn}\, L^1| + 6\ \mathrm{Mod}(L^2, L^1) + 6.$$

Proof. Suppose that for the transformation corresponding to the inverse of an elementary one, some Ω-domain Ω' splits into consecutive domains Ω_{i-1}, Ω_i, and Ω_{i+1}, where, by definition, the middle domain, Ω_i, is pure empty.

As in the proof of Lemma 1.3, let us consider two possibilities: either during the elementary transformation the domains Ω_{i-1} and Ω_{i+1} are glued together and Ω_i disappears, or the domains Ω_{i-1} and Ω_i disappear. Note that in the both cases, if the domains Ω_{i-1} and Ω_{i+1} are empty, then the domain Ω' is also empty, and, thus, the elementary transformation does not change the AB-module. Now, suppose that one of the domains Ω_{i-1} and Ω_{i+1} is not empty.

The three domains Ω_{i-1}, Ω_i, and Ω_{i+1} are transformed to the one domain Ω'. Note that either Ω' contains the both Ω_{i-1} and Ω_{i+1} (the first case), or Ω' is contained in one of Ω_{i-1} or Ω_{i+1} (the second case). In the both cases the number of $A-$ and B-domains does not increase.

Now, suppose that a polygonal line $\bar{L}^2$ is a reduced polygonal line obtained from L^2. The above observations imply the following lemma.

Lemma 1.6. *Under the above notations, we have*

$$\mathrm{ABMod}(\bar{L}^2, L^1) \le \mathrm{ABMod}(L^2, L^1).$$

By Corollary 1.2, the reduced polygonal line $\bar{L}^2$ has at most one F-domain, thus, according to Lemma 1.3 and Corollary 1.1, we have

$$\begin{aligned} |\operatorname{Ind}(L^2, L^1)| = |\operatorname{Ind}(\bar{L}^2, L^1)| &\le \mathrm{ABMod}(\bar{L}^2, L^1) + 1 \\ &\le \mathrm{ABMod}(L^2, L^1) + 1 = \mathrm{Mod}(L^2, L^1). \end{aligned}$$

Now, it remains to apply Theorem 1.1 and the following evident estimations:

$$|\alpha(L^2, L^1)| < 3 \quad \text{and} \quad |\beta(L^2, L^1)| < 3.$$

Corollary 1.3 is proven. □

Slightly more deep reasoning than in the proof of Corollary 1.3 provides the following results.

Corollary 1.4. *Let two polygonal lines L^1 and L^2 be in general position and join the same points A and B. Let $\bar{L}^2$ be a reduced polygonal line obtained from L^2. Consider the canonical partitions of L^2 and $\bar{L}^2$ with respect to L^1. Let p be the number of A-elements, and q be the number of B-elements of the partition of L^2. Then*

- *if in the partition of L^2 there exists an F-element, then*

$$|\operatorname{Ind}(L^2, L^1)| \le p + q + 1;$$

- *if in the both partitions of L^2 and $\bar{L}^2$ there is no an F-element, then*

$$|\operatorname{Ind}(L^2, L^1)| \le \max(p, q);$$

- *if in the partition of L^2 there is no an F-element, but there is an F-element in the partition of $\bar{L}^2$, then*

$$|\operatorname{Ind}(L^2, L^1)| \le p + q - 1. \qquad \square$$

Proof. The first item of the corollary is just the first statement of Corollary 1.3, and we present it here for reasons of completeness.

To prove the second item, it suffices to note that for elementary transformations of the both types described in the proof of Corollary 1.3, either a domain of A or B type disappears, or it transforms into an F-domain (for the transformation of the first type, when two domains of A and B types are glued into one F-domain), or it transforms into domain of the same type. Thus, the numbers of A- and B-domains in the partition of $\bar{L}^2$ do not exceed p and q, respectively. By Corollary 1.1 and 1.2, the word $W(\bar{L}^2, L^1)$ is of the form $A^x B^y$, where $xy \leq 0$, $|x| \leq p$, and $|y| \leq q$. So, we have proven this item also.

To complete the proof, note that if for an elementary transformation a new F-domain appears, then this transformation is of the first type, and as a result, two domains, one of A type and another of B type, have been glued into this F-domain. Thus, at this step the number of A-domains decreased by 1, and the same is true for the number of B-domains. Thus, in $\bar{L}^2$ the numbers of A- and B-domains are at most $p-1$ and $q-1$. Now, it remains to use the first item of the corollary, which completes the proof. $\square$

To conclude this section, we introduce some notions appearing in the case when one of L^1 and L^2 is closed, and another one is unclosed.

Let L^1 and L^2 be closed and unclosed polygonal lines, respectively, being in general position. Let $L^2 = \cup L_i^2$ be the canonical partition of L^2 with respect to L^1. Denote by W the domain bounded by L^1. Then, each of the inner elements L_i^2 lying outside W is called *a hat.* If we are also given that an ending point of L^2 belongs to L^1, and the corresponding ending element L_j^2 lies outside the W, then this ending element L_j^2 is also called a *hat.*

Let L_j^2 be a hat, and let A_j and B_j be the ending vertices of L_j^2. The points A_j and B_j partition the closed polygonal line L^1 into two components, say L' and L''. Denote by W' and W'' the corresponding domains bounded by the following pairs of the polygonal lines: (L_j^2, L'), and (L_j^2, L''). It is clear that one of the constructed domains is contained into another one. To be definite, suppose that $W' \subset W''$.

Definition. The interior of the domain W' is called *the hat corresponding to L_j^2*, or, *the H-domain for L_j^2*, and is denoted by $H(L_j^2)$. The polygonal line L' is called *the base* of the both L_j^2 and $H(L_j^2)$ and is denoted by $b(L_j^2)$.

It is clear that $\partial H(L_j^2) = L_j^2 \cup b(L_j^2)$.

Consider a special case when the polygonal line L^1 is generated by a convexity level. Namely, let M be a finite set of points of the plane, and M^t be its tth convexity level. Denote by σ^t the convex hull of M^t, and by W^t the boundary of σ^t, that is $W^t = \partial\sigma^t$. We put $\sigma = \sigma^1$, and $W = W^1$.

Let $L^2 \subset \sigma$ be an unclosed polygonal line being in general position with each W^t. Just that case we will consider in the proof of The Main Theorem.

Definition. A hat that the L^2 forms with respect to W^t will be called *a t-hat.*

Let L_i^2 be a t-hat. Suppose that L_i^2 intersects $W \setminus \sigma^s$ but does not intersect $W \setminus \sigma^{s-1}$ for some integer s. Then, evidently, $t \geq s$, and the polygonal line L_i^2 has an s-hat but has no $(s-1)$-hats.

Definition. Under the above assumptions, each s-hat is called *the top* of the hat L_i^2. Also, the number s is called *the index of the hat L_i^2* and is denoted by $\operatorname{ind}(L_i^2)$.

Remark. Note that if L_i^2 is a hat, and L' is some its top, then it is not necessary that $H(L_i^2) \supset H(L')$. However, the following proposition holds.

Proposition 1.3. *Let L_i^2 and L_j^2 be two t-hats, and $H(L_i^2) \subset H(L_j^2)$. Suppose also that the both hats have the same index s. Then for each top L_i' of L_i^2 there exists a top L_j' of L_j^2 such that $H(L_i') \subset H(L_j')$.*

Proof. Consider an arbitrary top L_i' of L_i^2. Since $L_i' \subset H(L_j^2)$, there exists a top L_j' of L_j^2 such that $L_i' \subset H(L_j')$. This immediately implies that $H(L_i') \subset H(L_j')$. The proposition is proven. □

Now, suppose that the polygonal line L^1 bounds a convex domain, and let the both ending points A and B of L^2 lie on L^1. These points partition L^1 into two components L' and L''.

Corollary 1.5. *Under the above assumptions, we have*

$$|\operatorname{tn} L^2| < 3\,|\operatorname{Ind}(L^2, L') + \operatorname{Ind}(L^2, L'')| + 6.$$

Proof. By Theorem 1.1,

$$\operatorname{tn} L^2 = \operatorname{tn} L' + 6 \operatorname{Ind}(L^2, L') - \alpha(L^2, L') - \beta(L^2, L'),$$
$$\operatorname{tn} L^2 = \operatorname{tn} L'' + 6 \operatorname{Ind}(L^2, L'') - \alpha(L^2, L'') - \beta(L^2, L''),$$

where the both L' and L'' are oriented from A to B.

Summing these two equations and observing that since L^1 and L^2 are in general position we have

$$|\alpha(L^2, L') + \alpha(L^2, L'')| < 3 \qquad \text{and} \qquad |\beta(L^2, L') + \beta(L^2, L'')| < 3,$$

and since L^1 bounds a convex domain we have

$$|\operatorname{tn} L' + \operatorname{tn} L''| \leq 6,$$

we obtain the statement of the corollary. □

2. Binary trees

Let us recall some important notions.

Definition. A subset Γ of the plane is called *a network* if it is connected and can be represented as union of a finite number of piecewise smooth embedded curves $\gamma_i\colon [0,1] \to \mathbb{R}^2$ such that the interior of any γ_i does not contain points from any other γ_j.

The curves γ_i are called *the edges* of the network Γ, and the points coinciding with ending points of the edges, that is, with points $\gamma_i(0)$ and $\gamma_i(1)$, are called *the vertices* of the network Γ. If a vertex V coincides with an ending point of an edge γ, then we say that V *is incident to* γ and that γ *is incident to* V. If a network Γ does not bound a compact domain, then it is called *a tree*. A network which is not a tree has *cycles*, that is in such a network there exist sequences of vertices $V_0, V_1, \ldots, V_{k-1}$ such that each pair of the consecutive vertices V_i and $V_{(i+1) \bmod k}$, $i = 0, \ldots, k-1$, are incident to the same edge e_i of the network Γ. The number of edges incident to a vertex is called *the degree* of the vertex. A tree is called *binary tree* or 2-*tree* if all its vertices are of degree 1 or 3. For an arbitrary network we define its *boundary* as an arbitrary subset of its vertex set. The last definition will be useful to define minimal networks.

Now, we define a *deformation of an edge* γ to be a piecewise smooth family γ_t, $t \in [0,1]$, of embedded piecewise smooth curves such that $\gamma_0 = \gamma$.

Let Γ be a network, and $\{e^i\}$ be the set of its edges. Let $\{e^i_t\}$ be a family of deformations of the edges e^i such that for any t the family $\{e^i_t\}$ is the set of edges of a network Γ_t, and the following property holds: some ending points of two edges e^i_t and e^j_t either coincide or do not coincide simultaneously for all $t \in [0,1]$. We call such a family Γ_t a *deformation of the network* Γ. Really, during such deformations we do not change the topology of the network.

We define *the length* of a network to be equal to the sum of lengths of all its edges. To define *local minimal network* or, simply *minimal network*, we proceed as follows. First, define *an admissible neighborhood* of a point P of a network Γ to be a closed neighborhood $U \subset \mathbb{R}^2$ of P such that

- U does not contain the vertices of Γ lying outside P,
- the intersection $\Gamma \cap \partial U$ consists of a finite number of points, and
- the intersection $\Gamma_U = \Gamma \cap U$ is a tree.

The network Γ_U will be referred as *the local network centered at* P. We define *the boundary* $\partial\Gamma_U$ of the local network to be the set $(\partial U \cap \Gamma) \cup (U \cap \partial\Gamma)$.

A network Γ is called *an absolute minimal network with the boundary* $\partial\Gamma$ if it has the least possible length among all networks with the same boundary. A network Γ is called *a local minimal network*, or, simply *a minimal network*, if for each its point there exists a local network Γ_U centered at this point which is absolute minimal one with the boundary $\partial\Gamma_U$. The following proposition is well-known, and is valid in much more general situation [**39**].

Proposition 2.1. *A network* Γ *is minimal if and only if for some of its representations the following properties take place:*

- *all edges are line segments,*
- *any pair of edges incident to the same vertex form an angle of at least* $120°$*, and*

- *all vertices of degree* 1 *and* 2 *are boundary vertices.*

In the present work we pay attention only to the minimal networks which are trees and which have no vertices of degree 2. Proposition 2.1 implies that such trees can have only the vertices of degree 1 or 3, thus they are 2-trees. Moreover, all edges of such 2-trees are line segments which meet at vertices of degree 3 by equal angles, and, thus, by angles of 120°. Such trees are called *minimal* 2*-trees.* Note that if Γ is a minimal 2-tree with the boundary M, and $P \in M$ is a vertex of degree 3, then Γ is also minimal 2-tree with the boundary $M \setminus \{P\}$. Thus, in what follows we always shall suppose that the boundary of a minimal 2-tree consists just of all the vertices of degree 1. Also, we always shall suppose that the boundary of an arbitrary 2-tree also consists just of the vertices of degree 1.

For investigation of minimal 2-trees the notion of twisting number turned out to be very useful. To define it, let us consider an arbitrary 2-tree Γ, choose some pair (a, b) of its edges, and consider the unique path γ in Γ joining these edges. Orient the path γ from a to b, and consider an arbitrary vertex P of degree 3, if any, lying inside γ (the vertex P is supposed to be non ending point of γ). Choose an admissible neighborhood U of P, then $U \cap \Gamma$ consists just of 3 points A_i, $i = 1, 2, 3$. Let a_1 be the first edge of γ incident to P, and a_2 be the last one. Let $A_i \in a_i$. Consider the arc δ of ∂U between A_1 and A_2 which contains A_3.

Definition. If the motion from A_1 to A_2 along the arc δ is clockwise, then we say that we *turn at* P *to the right side* and assign to P the number -1. Otherwise, we say that we *turn at* P *to the left side* and assign to P the number $+1$. The number assigned to P is called *the twisting at* P. Now, *the twisting number* $\operatorname{tw}(a, b)$ *of the ordered pair* (a, b) is the sum of twistings over all inner vertices of the path γ. We put $\operatorname{tw}(a, a) = 0$ by definition.

Definition. *The twisting number of a* 2*-tree* Γ is the maximum of twisting numbers $\operatorname{tw}(a, b)$ over all ordered pairs (a, b) of edges of Γ. We denote the twisting number of Γ by $\operatorname{tw}\Gamma$.

Let us observe that $\operatorname{tw}(a, b) = -\operatorname{tw}(b, a)$, and that $\operatorname{tw}(a, b) + \operatorname{tw}(b, c) =$

$\operatorname{tw}(a,c)$ for any edges a, b, and c lying on the same path in Γ. Also, if $\operatorname{tw}\Gamma = \operatorname{tw}(a,b)$ then both a and b are incident to vertices of degree 1 [**34**]. An edge incident to a vertex of degree 1, that is, to a boundary vertex, is called *a boundary edge.*

3. Some properties of minimal 2-trees

In this section we shall prove some important properties of minimal 2-trees which will be useful below.

Let Γ be a 2-tree, and suppose that γ is an arbitrary path in Γ oriented in some way.

Definition. The path γ is called *a net geodesic* if for any inner edge e of γ the twistings at the vertices incident to e have opposite signs.

Let P be a vertex of a 2-tree Γ, and e be an edge of Γ incident to P. Consider a maximal net geodesic starting at P and containing the edge e. We call such net geodesic *a net geodesic ray emitting from P in the direction of e.* It is clear that the other ending point of the net geodesic ray γ is a boundary point of Γ.

Now, let Γ be a minimal 2-tree, P be an arbitrary vertex of Γ, and e be an edge of Γ incident to P. Consider an angle α of $120°$ with the vertex at P such that the ray ℓ starting at P and containing the edge e is the bisector of α. The ray ℓ partitions the angle α into two angles. Each of these two angles is called *a characteristic wedge of the pair (P,e)* and is denoted by $\operatorname{wedge}^i(P,e)$, $i=1,2$. The following proposition is an immediate consequence of definition of the net geodesic ray.

Proposition 3.1. *Let P and e be a vertex and an edge of a minimal 2-tree Γ such that e is incident to P. Each characteristic wedge $\operatorname{wedge}^i(P,e)$ contains a net geodesic ray emitting from P in the direction of e, and each net geodesic ray emitting from P in the direction of e is contained in one of the characteristic wedges $\operatorname{wedge}^i(P,e)$. In particular, each characteristic wedge $\operatorname{wedge}^i(P,e)$ contains a boundary vertex of Γ distinct from P.*

Proposition 3.2. *Let Γ be a minimal 2-tree spanning a set M, L be a path* *in* Γ, *and* W *be a convex polygon bounded by* L'. *Suppose that* L *and* L' *are in general position. Consider the canonical partition* $L = \cup L_i$ *of* L *with respect to* L', *and let* L_j *be a hat. If the corresponding* H*-domain* $H(L_j)$ *contains a point of the path* L, *then* $H(L_j)$ *also contains a point from* M.

Proof. Suppose otherwise, that is a point P of L belong to $H(L_j)$ but $H(L_j) \cap M = \varnothing$. If P is not a vertex of Γ, then consider the edge e of Γ passing through P. Since W is convex, and $P \notin W$, then one of the vertices of Γ incident to e also does not belong to W. But the edge e can not cross L_j since in the opposite case we obtain a cycle in Γ, and that is impossible. Thus, this vertex belongs to $H(L_j)$, and, therefore, we can always assume that P is a vertex of Γ. Since such P does not belong to M, then it is Steiner point of Γ.

Now, let e_i, $i = 1, 2, 3$, be three edges of Γ incident to P. Consider six characteristic wedges $\mathrm{wedge}^j(P, e_i)$. Clear that one of this wedges does not intersect W (it is so, since W is convex). By Proposition 3.1, each of these wedges contains a net geodesic ray emitting from P. Let wedge' be the wedge which does not intersect W, and γ be a net geodesic ray emitting from P which is contained in wedge'. It is clear that such γ either finishes in $H(L_j)$, or intersects L_j. But the both cases are impossible, the first one since $M \cap H(L_j) = \varnothing$, and the second one since Γ has no cycles. Proposition 3.2 is proven. □

Corollary 3.1. *Let* Γ *be a minimal* 2*-tree spanning a set* M, *and* L *be a path in* Γ. *Let* L' *be a hat of* L *with respect to some convexity level of* M, *and* $\hat{L}'$ *be a top of* L'. *Then we have*

$$H(\hat{L}') \cap L = \varnothing.$$

Corollary 3.2. *Let* Γ *be a minimal* 2*-tree spanning a set* M, *and* L *be a path in* Γ. *Suppose that* L *has two hats* L' *and* L'' *with respect to some convexity level of* M *such that* $H(L') \subset H(L'')$. *Then we have*

$$\mathrm{ind}(L') \neq \mathrm{ind}(L'').$$

In particular, if $L^1, \dots, L^m$ are some hats of L such that

$$H(L^1) \subset \cdots \subset H(L^m),$$

and L^1 is t-hat, then $m < t$.

Proof. To prove the first part of the corollary, suppose otherwise, that is the indices of the both L' and L'' are equal to each other. Let $\hat{L}'$ be a top of L'. Then, by Proposition 1.3, there exists a top $\hat{L}''$ of L'' such that $H(\hat{L}') \subset H(\hat{L}'')$, thus $\hat{L}' \subset H(\hat{L}'')$. But the last statement contradicts Corollary 3.1.

The second part of the corollary immediately follows from the first one. Corollary 3.2 is proven. □

We shall use this results in the proof of The Main Theorem.

4. General position of a minimal 2-tree

Let Γ be a minimal 2-tree spanning a finite set M of points of the plane.

Definition. We say that Γ *is in general position* if all its edges are in general position with respect to each line segment connecting any pair of vertices of M.

Theorem 4.1. *For any minimal 2-tree Γ which spans a set M consisting of at least three points of the plane and for any $\varepsilon > 0$ there exists a perturbation of M such that for the perturbed set M' the following properties hold:*

- *the distance between each perturbed vertex $m' \in M'$ and its initial position $m \in M$ is less than ε;*
- *the partition of M into convexity levels is preserved during the perturbation;*
- *this perturbation of M can be extended to the perturbation of the network Γ through the class of minimal 2-trees;*
- *the resulting minimal 2-tree Γ' is in general position.*

Proof. The fact that an arbitrary sufficiently small perturbation of the set M can be extended to the deformation of the minimal network Γ through the class of minimal networks and, of course, without change of the topology of Γ, follows from the Melzak's algorithm of minimal 2-

trees construction. Thus, among such perturbations there exists a perturbation which preserves the partition of M into the convexity levels, and, moreover, in the perturbed M all the convexity levels are polygons without angles of π.

So, it remains to show that there exist sufficiently small perturbations such that the resulting Γ is in general position. To prove this, note first that by means of sufficiently small perturbation we can obtain from M a new set, which we again denote by M, such that the angles between directions of an arbitrary pair of line segments joining some points of M do not equal to $\pi k/3$ for any integer k. Thus, for such set M there can exist at most one segment parallel to an edge of the deformed network Γ which we again denote by the same letter Γ.

Suppose that such segment does exist, and denote it by a. Since the set M consists of at least 3 points, then there exists a point $P \in M$ which is not an ending point of a. Recall that any Simpson line for Γ whose direction coincides with the directions of the edges of Γ up to $\pi/3$ has the direction of the form

$$\sum_{P_j \in M, P_j \neq P} P_j \exp(\sqrt{-1}k_j\pi/3) + P\exp(\sqrt{-1}k\pi/3)$$

for some integer k_j and k. Here we represent the points of M as the corresponding complex numbers.

Since the multiplication on $\exp(\sqrt{-1}k\pi/3)$ is an isomorphism of the plane, then there exist small perturbations of P changing the direction of the Simpson line as slightly as we want. Thus, we can perturb the set M in such a way that the directions of all segments joining the points of M (including the segment a) differ from the directions of the edges of the deformed Γ. Again, we denote the result of the deformation by the same letters.

Now, it remains to show that we can remove the Steiner points of the network Γ from the segments joining the points of M. To do that, let us remove the Steiner vertices point-by-point. Evidently, it suffices to prove that we can remove one Steiner point.

To do that, denote by P a Steiner point which turned out lie in a segment a joining points of M, and denote by e an edge of Γ incident to P. Let us cut the network Γ along its edge e to obtain two networks Γ' and Γ'', and partition the set M into two components, say M' and M'', each of which consists of the vertices of Γ' and Γ'', respectively. To be definite, suppose that M' contains a point which is not an ending point of a.

Recall that for any edge of Γ there exists a Simpson line containing this edge. Denote by L such line containing e. Recall also that, by Melzak's algorithm, we can construct, using only the points from M'', a regular triangle T, and, using only the points from M', we can construct a point R, such that

- the Simpson line L is a segment QR where Q is one of the vertices of the triangle T;
- $P \in S^1 \cap L$, where S^1 is the circle circumscribed around the regular triangle T.

Since

- the position of vertices of the regular triangle T depends only on points of M'',
- small deformations of any point of M can change the direction of L (as we have proven above), and
- since a and L are not parallel, $P \in S^1 \cap L$, and during deformations of a point from M' which is not ending point of a the circle S^1 and the segment a remain fixed,

there exists a small perturbation of a vertex from M' which removes the point P from a. Theorem 4.1 is proven. □

5. Proof of The Main Theorem

Recall the main notations. Let Γ be a minimal 2-tree spanning a set M of points of the plane. Suppose that the set M has k convexity levels. We should prove that $\operatorname{tw}\Gamma \le 12(k-1)+5$.

Let a and b be some edges of Γ such that $\operatorname{tw}\Gamma = \operatorname{tw}(a,b)$. Recall that such a and b have to be boundary edges. Denote by L the path in Γ joining a and b, and orient the path L from a to b. Note that

$\operatorname{tn} L = \operatorname{tw} \Gamma$. Denote by A and B the boundary vertices of L incident to a and b, respectively. Note that the both A and B belong to M.

Let M^t denote the tth convexity level. Let $A \in M^p$, $B \in M^q$, and suppose that $p \leq q$.

Let σ^t be the convex hull of M^t, and W^t be the boundary of σ^t, $\partial \sigma^t = W^t$. We abbreviate $\sigma = \sigma^1$, and $W = W^1$.

Put $S^t = \sigma^t \setminus \sigma^{t+1}$. If $\sigma^{t+1} \neq \varnothing$ then S^t is a doubly connected set bounded by W^t and W^{t+1}, and such that $W^t \subset S^t$, but $W^{t+1} \cap S^t = \varnothing$.

Note that, by Theorem 4.1, we can assume without loss of generality that the network Γ is in general position. In what follows, we always shall suppose that Γ possesses this property.

Now, we consider two possible cases: $p = q$ and $p < q$. Let us start with the first one.

5.1. The case $p = q$

Since the case $p = q = 1$ was proven in [**36**], we suppose that $p \geq 2$.

Denote by W_1 and W_2 the parts of W^p into which the points A and B partition W^p. To start with, consider the canonical partition $L = \cup L^i$ of the path L with respect to W^p. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be all p-hats of L whose bases contain the points A and B, respectively. Since the bases of the elements of each $\mathcal{H}_A$ and $\mathcal{H}_B$ are ordered by inclusion, the same is true for the H-domains corresponding to these elements. Then, Corollary 3.2 implies the following result.

Proposition 5.1. *Each of the collections $\mathcal{H}_A$ and $\mathcal{H}_B$ contains at most $p - 1$ elements.*

Note that all other p-hats of L which belong neither to $\mathcal{H}_A$, nor to $\mathcal{H}_B$, are empty elements with respect to either W_1 or W_2.

Also, let us remark that each inner nonempty element with respect to either W_1 or W_2 contains a t-hat lying in $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$, and that different such nonempty elements generate different hats from $\mathcal{H}$. Using this information, we can obtain obvious estimations for $\operatorname{Ind}(L, W_i)$, but, it turns out, the direct use of the estimations gives rather rough estimation for the sum $|\operatorname{Ind}(L, W_1) + \operatorname{Ind}(L, W_2)|$ as we need to complete the proof in this case. So, let us make some preliminary preparations.

Consider an arbitrary hat corresponding to an empty domain Ω with respect to W_i. We call such hats *empty hats.* Now, using the same arguments as in the proof of Theorem 1.1, we can find a pure empty domain $\Omega' \subset \Omega$. Evidently, Ω' is also an empty hat with respect to W_i. Applying an elementary transformation to Ω', we reconstruct L in such a way that the resulting polygonal line has one less hat and one less empty domain with respect to the both W_1 and W_2. Note that during the elementary transformation we change neither the hats from $\mathcal{H}$, nor the directions of the ending edges of L. Iterating this process until all empty hats will be deleted, and using the fact that the turning does not change during such transformations, we obtain the following result.

Proposition 5.2. *There exists a deformation of the polygonal line L fixed on all nonempty hats and preserving the directions of the ending edges of L such that the resulting polygonal line does not have empty hats. Clearly, the turning of L also is preserved during the deformation.*

Thus, we can suppose that L has no empty hats. Such polygonal lines have the following important property.

Proposition 5.3. *If L has no empty hats, then for each A and B at least one of the corresponding ending domains (with respect to W_1 and W_2) contains a hat from $\mathcal{H}$.*

Now, denote by n_i^A the number of ending A-elements with respect to W_i which do not contain a hat from $\mathcal{H}_A$, by n_i^B the number of ending B-elements with respect to W_i which do not contain a hat from $\mathcal{H}_B$, and put $n_i = n_i^A + n_i^B$. Note that $n_i^A \leq 1$ and $n_i^B \leq 1$. Proposition 5.3 implies the following result.

Corollary 5.1. *Under the above notations, we have $n_1 + n_2 \leq 2$.*

Now, consider all possible types of partitions of the polygonal line L with respect to W_i.

(1) Let the partition of L with respect to W_i contain an F-element. Then, by Proposition 5.1, the total number of A- and B-elements does

not exceed $2(p-2)+n_i$, thus

$$|\operatorname{Ind}(L, W_i)| \leq 2(p-2)+n_i+1 = 2p-3+n_i.$$

(2) Let the partition do not contain an F-element. Then the numbers of A-elements and B-elements do not exceed $(p-1)+\max(n_i^A, n_i^B)$. Here we have two possibilities.

(2a) After a reduction with respect to W_i the canonical partition of the reduced L does not contain an F-element. Then, by Corollary 1.4,

$$|\operatorname{Ind}(L, W_i)| \leq p-1+\max(n_i^A, n_i^B) \leq 2p-3+\max(n_i^A, n_i^B)$$

for $p \geq 2$.

(2b) After a reduction with respect to W_i, the canonical partition of the reduced L contains an F-element, then, by Corollary 1.4,

$$|\operatorname{Ind}(L, W_i)| \leq p-1+n_i^A+p-1+n_i^B-1 \leq 2p-3+n_i.$$

Recall that, by Corollary 5.1, $n_1+n_2 \leq 2$. Since $\max(n_i^A, n_i^B) \leq 1$, then $\max(n_1^A, n_1^B)+\max(n_2^A, n_2^B) \leq 2$. Also, by Proposition 5.3, for $i \neq j$

if $\max(n_i^A, n_i^B) = 1$, then $n_j \leq 1$, and if $\max(n_i^A, n_i^B) = 0$, then $n_j \leq 2$,

therefore, $n_i + \max(n_j^A, n_j^B) \leq 2$. Thus, we obtain

$$|\operatorname{Ind}(L, W_1)+\operatorname{Ind}(L, W_2)| \leq 2(2p-3)+2 = 4(p-1).$$

Now, by Corollary 1.5, we have

$$|\operatorname{tn} L| < 3\,|\operatorname{Ind}(L, W_1)+\operatorname{Ind}(L, W_2)|+6 \leq 3*4(p-1)+6 = 12(p-1)+6.$$

Since the turning of a path in a minimal 2-tree is an integer number, we have proven The Main Theorem in this case.

5.2 The case $p < q$

Now, let $p < q$. Partition the path L into the following fragments. The first fragment, L_1, is the smallest closed connected part of L starting at the point A and such that $L \setminus L_1 \subset \sigma^p$. Put $A_1 = A$, and denote the other ending vertex of L_1 by B_1.

The second part of L, L_2, is the smallest closed connected part of L such that $L \setminus \cup_{j=1}^2 L_j$ is contained in σ^{p+1}. It is clear that B_1 is an ending point of L_2. We put $A_2 = B_1$, and we denote the other ending vertex of L_2 by B_2.

Further, for $i > 1$ such that $p+i-1 \le q$ we define L_i as the smallest closed connected part of L such that $L \setminus \cup_{j=1}^{i} L_j$ is contained in σ^{p+i-1}. Again, B_{i-1} is an ending vertex of L_i. We put $A_i = B_{i-1}$ and denote by B_i the other ending vertex of L_i.

The last fragment of the partition, L_{q-p+2}, if any, is the closure of the remained part of L.

Thus, we have partition L into fragments $L_1, L_2, \dots, L_{q-p+2}$, such that each L_i spans the points A_i and B_i, $A_1, B_1 \in W^p$, $A_i = B_{i-1} \in W^{p+i-2}$, and $B_i \in W^{p+i-1}$ for $2 \le i \le q-p+1$, and the both A_{q-p+2} and B_{q-p+2}, if any, belong to W^q. For each L_i denote its edge incident to A_i by a_i, and the edge incident to B_i by b_i.

We estimate the turning of L_1 using the results of the previous subsection. Thus, $|\operatorname{tn} L_1| \le 12(p-1)+5$. Further, we shall estimate $\operatorname{tn} L_i$ for $2 \le i \le q-p+1$ in the following proposition.

Proposition 5.4. *If $2 \le i \le q-p+1$, then $|\operatorname{tn} L_i| \le 9$.*

At last, we shall prove

Proposition 5.5. *If the part L_{q-p+2} is not empty, then*

$$|\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2}| \le 12.$$

Clear that Propositions 5.4 and 5.5 complete the proof of The Main Theorem.

Proof of Proposition 5.4. To start with, let us notice that if L_i consists of less than 4 edges, then the proposition holds. Thus, suppose that L_i has at least 4 edges. Orient L_i from A_i to B_i, and denote the last three consecutive edges by b'', b', and b_i. Let the vertex incident to b'' and b' be denoted by B'', and the vertex incident to b' and b_i be denoted by B'. Also, denote by c'' and c' the edges of the network Γ incident to B'' and B', respectively, and not belonging to L.

Denote by γ_1 and γ_2 two possible net geodesic rays emitting from B' in the direction of c'. Of course, they can coincide with each other. Consider the first intersection point C'_j of the net geodesic ray γ_j with polygons W^m. It is clear that C'_j belongs either to W^{p+i-2}, or to W^{p+i-1}. If C'_j belongs to W^l, we say that the net geodesic ray γ_j *comes first to* W^l.

There exist two possibilities: either one of γ_j, say γ_1, comes first to W^{p+i-2}, or the both γ_j come first to W^{p+i-1}.

Consider the first case. Now, denote again by γ_1 the part of γ_1 between B' and C'_1. Orient γ_1 from B' to C'_1, and denote by c'_1 the last edge of γ_1.

The following lemma is proven in [**36**].

Lemma 5.1. *Let W be a convex polygon, and let L be a path which joins two points $A \in W$ and $B \in W$, lies in the domain bounded by W, and is in general position with W. Denote by a and b the ending edges of L incident to A and B, respectively. Then, we have $|\operatorname{tw}(a,b)| < 6$. In particular, if L is a path in a minimal 2-tree, then $|\operatorname{tw}(a,b)| \le 5$.*

Now, denote by γ'_1 the path in Γ between A_i and C'_1. This polygonal line lies in σ^{p+i-2}, is in general position with $W^{p+i-2} = \partial\sigma^{p+i-2}$, and joins two points lying on W^{p+i-2}. Thus, by Lemma 5.1, $|\operatorname{tw}(a_i, c'_1)| \le 5$. By additivity of twisting number along a path, we have

$$\operatorname{tw}(a_i, c'_1) = \operatorname{tw}(a_i, b') + \operatorname{tw}(b', c') + \operatorname{tw}(c', c'_1) = \operatorname{tw}(a_i, b') + \operatorname{tw}(b', c') + \operatorname{tn}\gamma_1.$$

But, since γ_1 is a part of a net geodesic, $|\operatorname{tn}\gamma_1| \le 1$. Thus, we have

$$|\operatorname{tw}(a_i, b')| = |\operatorname{tw}(a_i, c'_1) - \operatorname{tw}(b', c') - \operatorname{tn}\gamma_1| \le 5 + 2 = 7,$$

and, therefore, $|\operatorname{tw}(a_i, b_i)| \le 8$.

Consider the second case, that is when the both γ_i come first to W^{p+i-1}. Let σ' be the least domain between γ_1, γ_2, and W^{p+i-1}. Let C' be the vertex of c' distinct from B'. Prolong the edge c' through the vertex C'. Clear that either the edge c' comes first to W^{p+i-1}, or its prolongation comes first into σ', and at some instant comes to $\partial\sigma' \cap W^{p+i-1}$. In other words, either the edge c' or its prolongation comes first to W^{p+i-1}, and, evidently, comes there transversely.

Thus, we have obtained that the rays which start at B', go in the directions of the edges b_i and c', and form an angle, say α, of $120°$, come first to W^{p+i-1}. This immediately implies that the both characteristics wedges $\mathrm{wedge}^j(B', b')$ do not intersect W^{p+i-1}, since the sides of the angle $\cup_{i=1}^2 \mathrm{wedge}^j(B', b')$ are the rays with the same origin B' but of the opposite directions to the sides of the angle α. Thus, both of the net geodesic rays emitting from B' in the direction of b' come first to W^{p+i-2} by the property of characteristic wedges, see Proposition 3.1. It is clear that one of the latter net geodesic rays contains a net geodesic ray emitted from B'' in the direction of c''. So, applying the same reasoning to this net geodesic ray as to γ_1, we obtain the new estimation $|\,\mathrm{tw}(a_i, b_i)| \le 9$, since between the edges b'' and b_i there are now two vertices instead of one vertex between b' and b_i. The Proposition 5.4 is proven.

Thus, if the part L_{q-p+2} is empty, then The Main Theorem is proven. So, suppose that L_{q-p+2} is not empty. Then, the above reasoning can be slightly generalized to obtain the desired estimation on the sum of indices of the last two L_j.

Proof of Proposition 5.5. Again, let us suppose that L_{q-p+1} consists of at least 4 edges. We shall use the same notations as in the proof of Proposition 5.4 for the fragment L_{q-p+1}. However, now we consider three possibilities: either the both γ_i come first to W^{q-1}, or one of them comes first to W^{q-1} but the other one to W^q, or the both γ_i come first to W^q.

As above, for the part γ_j' of Γ between the points A_{q-p+1} and C_j' we obtain the following estimation:

$$\begin{aligned}\mathrm{tw}(a_{q-p+1}, c_j') &= \mathrm{tw}(a_{q-p+1}, b') + \mathrm{tw}(b', c') + \mathrm{tw}(c', c_j')\\ &= \mathrm{tw}(a_{q-p+1}, b') + \mathrm{tw}(b', c') + \mathrm{tn}\,\gamma_j.\end{aligned}$$

Consider the first case. Observe that, since the both γ_j are emitted from the same vertex B' in the same direction c', for one of then, say for γ_1, the turning either equal 0, or have opposite sign to $\mathrm{tw}(b', c')$. Thus, we obtain

$$|\,\mathrm{tw}(a_{q-p+1}, b')| = |\,\mathrm{tw}(a_{q-p+1}, c_1') - \mathrm{tw}(b', c') - \mathrm{tn}\,\gamma_1| \le 5 + 1 = 6,$$

and, therefore, $|\,\mathrm{tw}(a_{q-p+1}, b_{q-p+1})| \le 7$.

By Lemma 5.1, $|\operatorname{tw}(a_{q-p+2}, b_{q-p+2})| \leq 5$, therefore, we get

$$|\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2}| \leq 12.$$

This case is proven.

For the last two cases we need the following result. First, to be definite, let us suppose that $\operatorname{tn} L \geq 0$ (if it is not so, we can make reflection through a line and apply the same reasoning). So, we shall estimate the turning of L from above.

For the rest of the proof we need the following result.

Lemma 5.2. *Let W be a convex polygon oriented anticlockwise, and let L be a path joining two points $A \in W$ and $B \in W$. Suppose that L lies in the domain bounded by W, and that L and W are in general position (the points A and B are supposed to be fixed). Orient the path L from A to B, and denote by a and b the first and the last edges of L. Let α be the twisting from the vector-edge of W containing A to the vector-edge a. Then, we have*

$$\operatorname{tw}(a, b) < 6 - \alpha.$$

Proof. The path L partitions the domain bounded by W into two domains. Denote by σ that of these two domains for which we pass L from A to B during the anticlockwise motion along $\partial\sigma$. Note that the twisting at A during this motion equals α. Let β be the twisting at B, and denote by x the turning along $\partial\sigma \cap W$. Thus, by Proposition 1.1, we have

$$\alpha + \operatorname{tw}(a, b) + \beta + x = 6.$$

Since W is convex polygon, we have $x \geq 0$. Also, evidently that $\beta > 0$. These observations complete the proof. □

Corollary 5.2. *Under the assumptions of Lemma 5.2, suppose also that L is a part of a minimal 2-tree. Then*

- *if* $\alpha \geq 1$ *then* $\operatorname{tw}(a, b) \leq 4$;
- *if* $\alpha \geq 2$ *then* $\operatorname{tw}(a, b) \leq 3$.

We use these results to finish the proof. Let us orient W^q anticlockwise, the fragment L_{q-p+2}, as before, from A_{q-p+2} to B_{q-p+2}, and denote by α the twisting from the vector-edge of W_q containing the point A_{q-p+2} to the vector-edge a_{q-p+2}.

Now, consider the second case. Since one of γ_i, say γ_1, comes first to W^{q-1}, we obtain $|\operatorname{tw}(a_{q-p+1}, b')| \le 7$. Here we consider two more possibilities: either $\operatorname{tw}(b', b_{q-p+1}) = -1$, or $\operatorname{tw}(b', b_{q-p+1}) = 1$. In the first case we have $\operatorname{tn} L_{q-p+1} \le 6$, thus,

$$\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2} \le 6 + 5 < 12.$$

In the second case we shall estimate the twisting α introduced above and apply Corollary 5.2. First, it is clear that in this case we have $\operatorname{tn} L_{q-p+1} \le 8$. Consider the least domain σ' between b_{q-p+1}, γ_2, and W^q. It is obvious that the prolongation of the edge b' through the vertex B' comes first to the domain σ', and after to $\partial\sigma' \cap W^q$, therefore, this prolongation comes first to W^q. This fact, together with the assumption $\operatorname{tw}(b', b_{q-p+1}) = 1$, imply that $\alpha > 1$ (the corresponding angle is more than $\pi/3$). Thus, by Corollary 5.2, we obtain

$$\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2} \le 8 + 4 = 12.$$

(Recall that we are estimating the turning of L from above.) Thus, this case is also proven.

Consider the last case, that is when the both γ_i come first to W^q. In this case we have $|\operatorname{tw}(a_{q-p+1}, b')| \le 8$. Again, consider the following two possibilities: $\operatorname{tw}(b', b_{q-p+1}) = -1$, or $\operatorname{tw}(b', b_{q-p+1}) = 1$. In the first case we obtain $\operatorname{tn} L_{q-p+1} \le 7$, and, thus,

$$\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2} \le 7 + 5 = 12.$$

In the second case we again shall use some estimations on the twisting α. First, we have $\operatorname{tn} L_{q-p+1} \le 9$. Now, consider the edge c', and recall that either this edge, or its prolongation through the vertex C' comes first to W^q. Evidently, this fact, together with the assumption $\operatorname{tw}(b', b_{q-p+1}) = 1$, imply that $\alpha > 2$, thus, by Corollary 5.2, we have $\operatorname{tn} L_{q-p+2} \le 3$, and again, we obtain

$$\operatorname{tn} L_{q-p+1} + \operatorname{tn} L_{q-p+2} \le 9 + 3 = 12.$$

Proposition 5.5, and, thus, The Main Theorem, are completely proven. □

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